Harder Area Problems

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1. Given that ABCD is a rectangle, as in Figure 1 .
Area of $\Delta ABP = a$,
Area of $\Delta PCQ = b$,
Area of $\Delta ADQ = c$,
find the area of ΔAPQ .

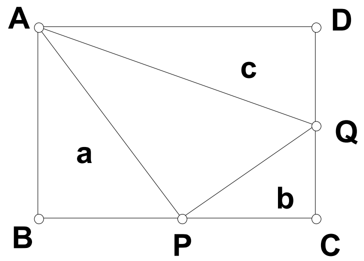


Figure 1

2. As in Figure 2, Q is a point inside ΔABC .
From Q , draw three lines parallel to sides of ΔABC .
These three lines divides the triangle into six portions.
If we knows the area of three parallelogram to be a, b, c .
Find the area of ΔABC .

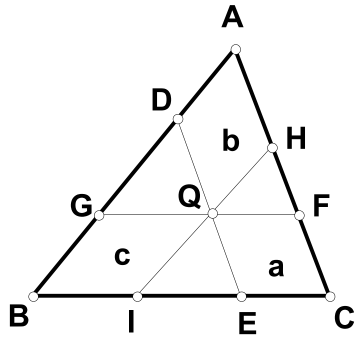


Figure 2

3. In Figure 3, BD : DC = 3 : 4 in ΔABC . F is a point on AD such that AF : FD = 2 : 5. CF is produced to meet AB at E.
Find AE : EB.

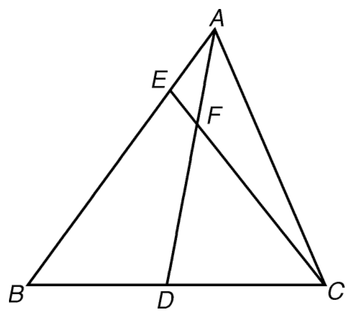


Figure 3

4. In Figure 4, CD and BE intersect at F in ΔABC ,
If the area of ΔBDF is 5, the area of ΔBCF is 10,
the area of ΔCEF is 8， find the area of ADFE.

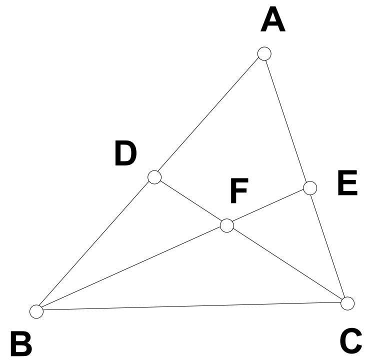


Figure 4

5. In Figure 5, ABCD is a square.
P is a point inside the square.
If PA = 3, PC = 7, PD = 5.
find the area of ABCD .

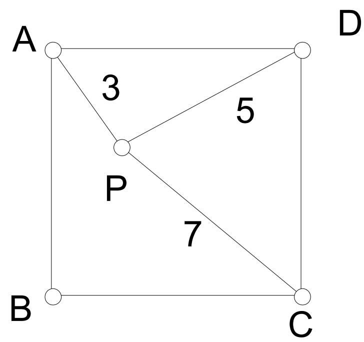
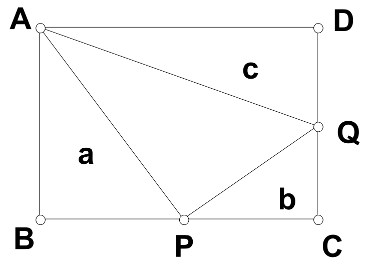


Figure 5

Solution

1. Let $AB = x$, $AD = y$, then $QC = x - \frac{2c}{y}$, $PC = y - \frac{2a}{x}$



Let $\Delta APQ = S$, we have : $\begin{cases} xy = a + b + c + S & (1) \\ \left(x - \frac{2c}{y}\right)\left(y - \frac{2a}{x}\right) = 2b & (2) \end{cases}$

From (2), $(xy - 2c)\ (xy - 2a) = 2bxy$

$(xy)^2 - 2(a + b + c)(xy) + 4ac = 0$

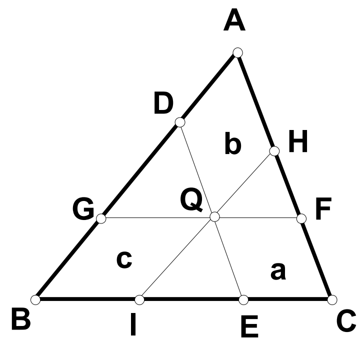
Since $xy > 0$, $xy = \frac{2(a+b+c) + \sqrt{4(a+b+c)^2 - 16ac}}{2}$

$= (a+b+c) + \sqrt{(a+b+c)^2 - 4ac}$ (3)

(3) ↓ (1), $S = \sqrt{(a+b+c)^2 - 4ac}$.

2. Let the areas of ΔQFH, ΔQIE, ΔGDQ be x, y, z respectively.

Then :



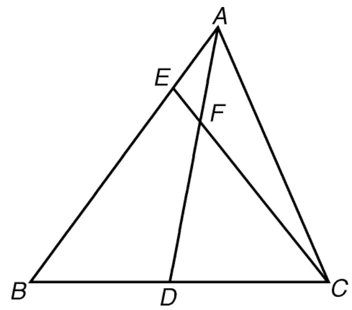
$\frac{y}{c} = \frac{IE}{2BI} = \frac{QE}{2DQ} = \frac{a}{2b}$, $\therefore y = \frac{ac}{2b}$

$\frac{x}{b} = \frac{HF}{2AH} = \frac{QF}{2GQ} = \frac{a}{2c}$, $\therefore x = \frac{ab}{2c}$

$\frac{z}{c} = \frac{GD}{2GB} = \frac{DQ}{2QE} = \frac{b}{2a}$, $\therefore z = \frac{bc}{2a}$

Area of $\Delta ABC = a + b + c + \frac{bc}{2a} + \frac{ac}{2b} + \frac{ab}{2c}$

3. AF : FD = 2 : 5, then for some constant k, we have



∴ Area of ΔAFC = 2k,

Area of ΔFDC = 5k.

BD : DC = 3 : 4, then

Area of ΔBDF : Area of ΔFDC = 3 : 4 .

Area of ΔBDF = $\frac{3}{4} \times 5k = \frac{15k}{4}$

Area of ΔBFC = $5k + \frac{15k}{4} = \frac{35k}{4}$.

AE : EB = Area of ΔAEC : Area of ΔBEC (1)

AE : EB = Area of ΔAEF : Area of ΔBEF (2)

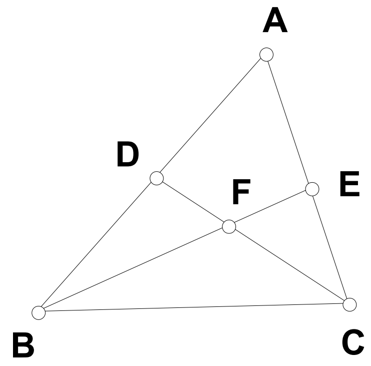
Since Area of ΔAFC = Area of ΔAEC – Area of ΔAEF

and Area of ΔBFC = Area of ΔAEF – Area of ΔBEF

we have AE : EB = Area of ΔAFC : Area of ΔBFC $= 2k : \frac{35k}{4} = \underline{\underline{8 : 35}}$

4. Join AF.

Let $\Delta ADF = x,\ \Delta AEF = y$



Then $\Delta ABF : \Delta AEF = BF : EF = \Delta BCF : \Delta CEF$.

$\therefore (5 + x) : y = 10 : 8$

$\therefore 4x - 5y + 20 = 0$ ….. (1)

Similarly $\Delta ADF : \Delta ACF = DF : CF = \Delta BDF : \Delta BCF$.

We have $x : (y + 8) = 5 : 10$

$\therefore 2x - y - 8 = 0$ …..(2)

Solving (1),(2) we get $x = 10, y = 12$.

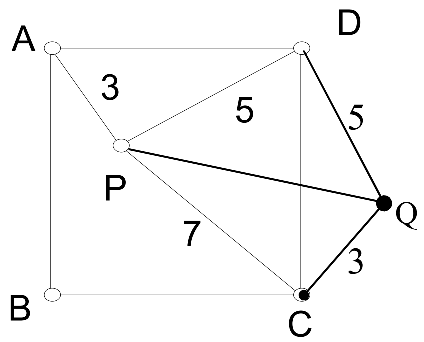
$\therefore$ The area of ADFE is $x + y = \underline{\underline{22}}$ square units

5. Rotate ΔAPD about D to get ΔDQD.

$\therefore \quad \Delta APD \equiv \Delta DQD$.

Then ΔPDQ is a right-angled isosceles triangle.

$\angle DPQ = 45^{o}$.



$PQ = \sqrt{5^2 + 5^2} = \sqrt{50}$

In ΔPQC, let $\angle QPC = \theta$.

Apply cosine law, we have

$$\cos\theta = \frac{\left(\sqrt{50}\right)^2 + 7^2 - 3^2}{2\left(\sqrt{50}\right)(7)} = \frac{50 + 49 - 9}{14\sqrt{50}} = \frac{90}{14\sqrt{50}} = \frac{90\sqrt{50}}{14 \times 50} = \frac{9\sqrt{2}}{14}$$

Also, $$\sin\theta = \sqrt{1 - \left(\frac{9\sqrt{2}}{14}\right)^2} = \frac{\sqrt{34}}{14}$$

Let the side of the square = x.

Apply cosine law to ΔPCD,

$$x^2 = 7^2 + 5^2 - 2 \times 7 \times 5\cos\left(45^\circ + \theta\right) = 49 + 25 - 70\left(\cos 45^\circ \cos\theta - \sin 45^\circ \sin\theta\right)$$

$$= 74 - 70\left(\frac{1}{\sqrt{2}} \times \frac{9\sqrt{2}}{14} - \frac{1}{\sqrt{2}} \times \frac{\sqrt{34}}{14}\right) = 74 - \frac{70}{14}\left(9 - \sqrt{17}\right) = 74 - 5\left(9 - \sqrt{17}\right) = \underline{\underline{29 + 5\sqrt{17}}}$$

Area of the square $= x^2 = \underline{\underline{29 + 5\sqrt{17}}}$